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A Symplectic Hamiltonian Derivation of Quasilocal Energy-Momentum for GR

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Abstract

The various roles of boundary terms in the gravitational Lagrangian and Hamiltonian are explored. A symplectic Hamiltonian-boundary-term approach is ideally suited for a large class of quasilocal energy-momentum expressions for general relativity. This approach provides a physical interpretation for many of the well-known gravitational energy-momentum expressions including all of the pseudotensors, associating each with unique boundary conditions. From this perspective we find that the pseudotensors of Einstein and Møller (which is closely related to Komar's superpotential) are especially natural, but the latter has certain shortcomings. Among the infinite possibilities, we found that there are really only two Hamiltonian-boundary-term quasilocal expressions which correspond to *covariant* boundary conditions; they are respectively of the Dirichlet or Neumann type. Our Dirichlet expression coincides with the expression recently obtained by Katz and coworkers using Noether arguments and a fixed background. A modification of their argument yields our Neumann expression.

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I. INTRODUCTION

Via their energy-momentum density, material sources generate gravitational fields. Sources interact with the gravitational field locally, hence they should be able exchange energy-momentum with the gravitational field — locally. From this physical conception we are led to expect the existence of a local density for gravitational energy-momentum. However the identification of a good localization for gravitational energy-momentum has turned out to be an outstanding fundamental problem. Standard techniques led only to various reference frame dependent complexes referred to as *pseudotensors*. This result can be understood in terms of the *equivalence principle* which implies that one can not detect any feature of the gravitational field at a point. Consequently, the whole idea has been criticized (see, e.g., [35], p 467) and the pseudotensor approach in particular has largely been abandoned.

A new idea: *quasilocal* (i.e., associated with a closed 2-surface) [38] has become widely accepted. Many quasilocal proposals have been considered (for the older works see [8]; more recent works are cited in [14,10]). Many well-known quasilocal expressions obtained by different approaches have been discussed in the literature; although they generally give different values [5], most seem to work well enough at least for certain physical situations. A number of criteria for selecting a good quasilocal expression (see, e.g., [15]), including good limits at spatial infinity, at future null infinity, to weak fields, and to flat spacetime have been advocated. Such requirements, however, have proved to be insufficient; in fact it has been noted that there still exist an infinite number of expressions satisfying the proposed criteria [6]. We infer that additional *principles* and *criteria* are very much needed to reduce and to parameterize, if not entirely eliminate, the ambiguity.

One might hope that there would exist a "best" gravitational energy-momentum expression which has either not yet been identified or at least not yet accorded widespread acceptance. On the other hand, it is well to keep in mind that there are physical situations where there is not one unique energy. One example is thermodynamics, wherein there are several energies (viz., internal, enthalpy, Gibbs and Helmholtz) corresponding to different choices of boundary conditions and independent variables; each one gives the relevant value of the energy for a particular physical situation. An even more appropriate example is electrostatics. It is well known that the work done in moving a system of charges and dielectrics differs depending on whether one holds the potential or the charge density fixed. We expect gravity to behave in a similar fashion: consequently *various definitions of gravitational energy may each be associated with their own unique boundary condition.* Fortunately, there exists a systematic technique, *symplectic analysis* [32], which, along with its associated control-response relations, can be used to identify the relationship between an energy-momentum expression and the associated boundary conditions.

Our approach to quasilocal energy-momentum is by way of the Hamiltonian formulation — essentially, we take energy to be given by the value of the Hamiltonian. The rationale goes back to Noether's work connecting symmetries and conserved currents: in particular energy-momentum is associated with translations in spacetime. The Hamiltonian is the Noether canonical generator of timelike displacements. The Hamiltonian for gravitating systems for a finite region of spacetime includes, in addition to a volume density term, a surface term which plays a key role. Its value will determine the quasilocal energy-momentum and,

through its variation, it governs the associated boundary conditions. In an earlier work [14] we presented our ideas, applied to rather general geometric gravity theories including Einstein's general relativity (GR), in terms of differential forms. However that technique is not so well known by many people interested in gravitational energy-momentum; moreover most people are mainly interested specifically in GR. Hence we present here a formulation of the application of our ideas to GR in the more traditional holonomic (coordinate basis) style. This will serve not only to make our ideas more widely accessible but will also facilitate comparison with the results obtained by other investigators, e.g., [7,18,26,8,43,39].

Elsewhere we have used our Hamiltonian-boundary-term approach to quasilocal quantities to show that all of the pseudotensors give well defined quasilocal energy-momentum values, each of which is associated with a particular choice of boundary conditions [10]. From the Hamiltonian boundary approach we find that two, the Einstein and Møller expressions, each arise naturally, although the latter has a few extra shortcomings. A more important failing however, in our opinion, is that none of the pseudotensors is associated with a truly *covariant* boundary condition.

For the required new *principle* and *criteria* to restrict the GR quasilocal energy-momentum expression we have advocated the *Hamiltonian boundary variation principle* and the criteria of *covariance*. For GR we found that there are only two *covariant* quasilocal expressions (both of which depend on the choice of a reference configuration on the boundary) which correspond, respectively, to Dirichlet and Neumann boundary conditions. At first we were surprised to learn that our Dirichlet expression coincides with an expression developed by Katz and coworkers [34,28,29] using a different approach based on a Noether type argument at the Lagrangian level along with a fixed global background geometry. With hindsight we now see that this agreement is related to the close connection between the Hamiltonian and Noether translation current. Here we show how to modify their argument to also obtain our Neumann quasilocal energy-momentum expression. Along the way we clarify the roles of the boundary (or total derivative) terms in the Lagrangian, the Hamiltonian and their respective variations.

II. THE SYMPLECTIC IDEA IN GENERAL RELATIVITY

In this section we outline the symplectic idea for Lagrangian and Hamiltonian formulations (here and elsewhere we are much influenced by Kijowski and coworkers [25,30–32]) for general relativity (GR), Einstein's theory of gravity (a detailed discussion for general geometric gravity theories, in terms of differential forms, appears in [14]). The simple and direct way to reveal the symplectic structure of a physical configuration is through the variation of the associated Lagrangian or Hamiltonian.

A. Lagrangian formulation

Let us first briefly review some features of the Lagrangian variational principle for classical field theories. For our purposes we found it convenient to use the first order formalism. For a field ϕ^A the first order Lagrangian scalar density has the form

$$\mathcal{L} = P^\mu_A \partial_\mu \phi^A - \Lambda(\phi^A, P^\mu_A). \tag{1}$$

The field equations (which in this formulation contain only first derivatives of the fields) are taken to be the Euler-Lagrange expressions implicitly determined by the variation:

$$\delta\mathcal{L} = \partial_\mu (P^\mu_A \delta\phi^A) + \frac{\delta\mathcal{L}}{\delta\phi^A}\delta\phi^A + \frac{\delta\mathcal{L}}{\delta P^\mu_A}\delta P^\mu_A. \tag{2}$$

The action is obtained by integrating the Lagrangian density over a spacetime region; the variation of the action is given by the integral of (2). When integrated over a spacetime region the total derivative term becomes a boundary term. Technically the variational derivatives of the action are well defined only if the boundary term in the variation vanishes. That requirement shows what must necessarily be held fixed on the boundary, this quantity is referred to as the "control variable". In this case it is the field ϕ^A, thus this Lagrangian is differentiable only on the space of fields for which ϕ is given a predetermined dependence on the boundary. The variation boundary term, moreover, has a certain *symplectic* form which connects the "control variable" with an associated "response variable" (in this case P^μ_A).

We are only concerned with actions which do not depend on the position except via the fields. Hence they have an invariance under local infinitesimal translations (i.e, diffeomorphisms) — which can be represented by Lie derivatives. Thus, for an arbitrary vector field N,

$$\pounds_N \mathcal{L} := \partial_\nu (N^\nu \mathcal{L}) \equiv \partial_\mu (P^\mu_A \pounds_N \phi^A) + \frac{\delta\mathcal{L}}{\delta\phi^A}\pounds_N\phi^A + \frac{\delta\mathcal{L}}{\delta P^\mu_A}\pounds_N P^\mu_A. \tag{3}$$

From this identity we conclude, by taking N^μ to have constant coefficients, that the *canonical energy-momentum* density,

$$T^\mu{}_\nu := P^\mu_A \partial_\nu \phi^A - \delta^\mu_\nu \mathcal{L}, \tag{4}$$

is conserved. More precisely its divergence is proportional to a combination of the field equations and hence vanishes "on shell". Note that the canonical energy-momentum density is not unique in the sense that we can add to it an expression which is automatically divergence free. Such an expression is necessarily of the form $\partial_\gamma U_\nu{}^{\mu\gamma}$ where $U_\nu{}^{\mu\gamma} \equiv -U_\nu{}^{\gamma\mu}$. This ambiguity allows one to adjust the zero of energy and has been exploited to find "improved" energy-momentum tensors such as the symmetrized one constructed by Belinfante [3] and Rosenfeld [41]. On the other hand, since we have also assumed invariance with non-constant N^μ, we are also requiring that (3) be identically satisfied for the terms proportional to ∂N (this is only possible if the list of dynamic variables includes certain geometric variables). In this way we discover that $T^\mu{}_\nu$ itself is linear in the field equations and thus vanishes "on shell". In other words the 'conservation law' is actually a differential identity connecting the field equations showing that they are not all independent, hence the evolution of the dynamical variables is underdetermined—a fact which is directly related to the local 'translational' gauge (i.e., diffeomorphism) freedom of the theory.

Now let us apply this analysis to gravity. There are several choices of variables and Lagrangians which can be used. Since we favor a first order approach, a natural geometric choice is to regard the metric and connection as independent fields. Even within this overall approach there are various options, in particular the metric degrees of freedom can alternately be encoded in terms of an orthonormal frame while the torsion free and metric

compatibility conditions can be imposed *a priori*, or enforced via Lagrange multipliers, or they can be obtained as dynamic field equations (this is easily done in the vacuum case which is all that we consider here). All of these approaches merit consideration. We have investigated many of the possible combinations; our preliminary conclusion is that they lead to essentially the same result [9]. Here we consider explicitly only one case which is relatively simple in the holonomic treatment.

The field equations of (vacuum) GR can be obtained from a first order variational principle using the Hilbert Lagrangian density in the so-called 'Palatini' form [44]

$$\mathcal{L}_H := \pi^{\mu\nu} R_{\mu\nu}(\Gamma). \tag{5}$$

Here we are using the contravariant metric density, defined by $\pi^{\mu\nu} := (2\kappa)^{-1}\sqrt{-g}g^{\mu\nu}$, where $\kappa := 8\pi G/c^4$, and the (symmetric, i.e., torsion free) connection coefficients, $\Gamma^{\mu}{}_{\alpha\beta}$, as independent variables (while our conventions are generally those of [35], our treatment can be compared with [30,31] in which the same variable combinations appear). The variation of the Lagrangian density, after the usual integration by parts, has the form

$$\delta\mathcal{L}_H = \frac{\delta\mathcal{L}}{\delta\pi}\delta\pi + \frac{\delta\mathcal{L}}{\delta\Gamma}\delta\Gamma + \partial_\gamma \left(\pi^{\beta\nu}\delta^{\gamma\alpha}_{\mu\nu}\ \delta\Gamma^{\mu}{}_{\alpha\beta}\right). \tag{6}$$

The variational derivatives will give the desired field equations: $R_{\mu\nu} = 0$, from the variation with respect to $\pi^{\mu\nu}$, and $D_\lambda \pi^{\mu\nu} = 0$ (equivalent to $D_\lambda g_{\mu\nu} = 0$), from the variation with respect to Γ. When integrated over a spacetime region, the total derivative term gives rise to a boundary term. This boundary term shows that the control variable is the connection and the response variable is linear in $\pi^{\mu\nu}$. The variational derivatives are well defined only if this boundary variation term vanishes [40]. For a finite region this means we must 'control' or 'hold fixed' (i.e., give as a prespecified function) Γ on the boundary. For an asymptotically flat region the connection vanishes asymptotically, nevertheless $\Gamma = 0$ is not a sufficient boundary condition. Since we must allow for variations with the generic spatial fall offs $\delta\pi \sim O(1/r)$, $\delta\Gamma \sim O(1/r^2)$, the Lagrangian boundary variation term will yield a finite result in the asymptotic limit. Formally the situation is then described by saying that, in this case, the variational derivatives of the action are not well defined on the full space of asymptotically flat metric and connection fields, but rather only on the subspace where we actually fix the specific asymptotic form of Γ. This 'problem' is closely related to the fact that the Hilbert Lagrangian density is asymptotically $O(1/r^3)$; consequently the action diverges for $r \to \infty$. The remedy is simple: adjust the Lagrangian density by adding a total derivative term.

For GR, an obvious alternative is the "first order" (in derivatives of the metric) Lagrangian density, initially introduced by Einstein, which can be easy obtained by adding a total derivative term to (5):

$$\begin{aligned}\mathcal{L}_E &:= \mathcal{L}_H + \partial_\gamma\left(\pi^{\beta\nu}\Gamma^{\mu}{}_{\alpha\beta}\delta^{\alpha\gamma}_{\mu\nu}\right)\\ &\equiv \pi^{\mu\nu}\left(\Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha} - \Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\beta}{}_{\alpha\beta}\right).\end{aligned} \tag{7}$$

The variation of the Einstein Lagrangian,

$$\delta\mathcal{L}_E = \ \text{(fields eq. terms)}\ + \partial_\gamma\left(\Gamma^{\mu}{}_{\alpha\beta}\delta^{\alpha\gamma}_{\mu\nu}\ \delta\pi^{\beta\nu}\right), \tag{8}$$

has the same field equation terms as before but a different boundary term which reflects an alternate symplectic structure and shows that the variable to be held fixed — the "control variable"— is now the contravariant metric density while the "response variable" is a certain combination of the connection. Now, for asymptotically flat fall offs, the Lagrangian boundary variation term does vanish asymptotically, so the variational derivatives are well defined on the space of all asymptotically flat fields (a related fact is that the Einstein action is finite).

However the big drawback is that we now have a response expression which is linear in Γ, a non-tensorial, reference frame dependent object (along with this the Lagrangian density itself is not covariant). The cure for this new 'problem' is to introduce a background (or reference) connection $\bar{\Gamma}$ (actually this is really only essential on the boundary) and define $\Delta\Gamma := \Gamma - \bar{\Gamma}$. The latter, being the difference between two connections, is a *covariant* quantity, which can be used in the Lagrangian density boundary (i.e., total derivative) term. The 'improved Einstein' action is now

$$\mathcal{L}_{IE} = \mathcal{L}_H + \partial_\gamma \left(\pi^{\beta\nu} \Delta\Gamma^\mu{}_{\alpha\beta} \delta^{\alpha\gamma}_{\mu\nu} \right). \tag{9}$$

The variation gives the same field equation terms but now has a *covariant* boundary-variation symplectic structure:

$$\delta\mathcal{L}_{IE} = \text{ (fields eq. terms) } + \partial_\gamma \left(\Delta\Gamma^\mu{}_{\alpha\beta} \delta^{\alpha\gamma}_{\mu\nu} \, \delta\pi^{\beta\nu} \right). \tag{10}$$

The Lagrangian boundary term does not, as is well known, affect the field equations. What it does affect is the boundary conditions implicit in the action. From another point of view, changing the action by a total derivative term amounts to a *canonical transformation*; in particular, as we saw in the cases considered, it is possible to interchange the role of 'coordinate' and 'momentum'. Thus the Lagrangian variational boundary term possesses important information: the *symplectic structure* representing the control–response relation of the system [32,45]. For instance, the symplectic structure in (6) shows that the connection is the control variable and the response is a certain combination of the metric.

B. Hamiltonian formulation

The energy of a gravitating system can be identified with the value of the Hamiltonian. However the Hamiltonian approach necessitates a splitting of spacetime at least to some extent. One constructs a $3+1$ foliation of spacetime by selecting a time function t such that the hypersurfaces, Σ_t, of constant t are space-like Cauchy surfaces. The standard Hamiltonian formulation for general relativity is the ADM representation (see.e.g., [1,23] and [35] Ch 21), in which 4-covariant objects are decomposed into various 3-covariant parts. In particular, the spacetime metric, $g_{\mu\nu}$, is decomposed into the form,

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -N^2 dt^2 + h_{ab}(dx^a + N^a dt)(dx^b + N^b dt), \end{aligned} \tag{11}$$

which depends on three spatially covariant parts: the *lapse function* N, the *shift vector* N^a and the spatial metric h_{ab}, induced on Σ_t. The associated Hamiltonian density is obtained from

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ab}} \dot{h}_{ab} - \mathcal{L}, \tag{12}$$

where $\dot{h}_{ab} \equiv \partial_t h_{ab} := \pounds_t h_{ab}$. Although this approach has led to much insight, it has the drawback that the resultant Hamiltonian formulation is only manifestly 3-dimensionally covariant.

We prefer to use a more "covariant" approach to the Hamiltonian density. To this end it is convenient to use the so-called 'Palatini' method of treating the metric and connection independently. (Our approach is in many ways similar to that of Kijowski, see, e.g. [31].) Let us note some general features. First of all, since we are concerned with localization, we shall want to find the Hamiltonian which can evolve a *finite* spatial region. To achieve our 'covariant' formulation, we represent the time evolution direction as a covariant 4-vector field N^μ. Quite generally the (4-covariant) Hamiltonian — which is essentially the Noether generator of translations (i.e., Lie derivatives) along N^μ — is given by the spatial integral over (the finite spatial hypersurface) Σ_t of a 4-covariant Hamiltonian density. In order to generate the Lie derivatives, the Hamiltonian density is necessarily linear in the time displacement vector field N^μ and its derivatives. Consequently it can be expanded into the form

$$\mathcal{H}^\mu(N) = N^\nu \mathcal{H}^\mu{}_\nu + \partial_\nu [\mathcal{B}^{\mu\nu}(N)], \tag{13}$$

where, it turns out that (at least for our representation) $\mathcal{B}^{\mu\nu}(N) \equiv -\mathcal{B}^{\nu\mu}(N)$.

On the other hand, beginning from the Lagrangian density, we can apply our Noether type argument to a translation along N^μ, (see [14]). Formally we then arrive at a conserved quantity (essentially the canonical energy-momentum density discussed earlier) which is actually just this same Hamiltonian density. From this analysis we learn a couple of important things. First, we find that ("on shell") the Hamiltonian density is necessarily conserved: $\partial_\mu \mathcal{H}^\mu(N) \equiv 0$. At this point we want to call attention to the fact that the possibility of adjusting the canonical energy-momentum density (4) by adding an automatically divergence free part exactly corresponds to adjusting $\mathcal{B}^{\mu\nu}$ in (13). Second, we find that $\mathcal{H}^\mu{}_\nu$ is linear in the field equations and thus vanishes "on shell". From this latter fact we conclude that the numerical *value* of the Hamiltonian is completely determined by the $\partial\mathcal{B}$ term, which, when integrated over a spatial region Σ_t, via the divergence theorem, gives rise to an integral of $\mathcal{B}$ over the 2-dimensional spatial boundary $\partial\Sigma_t$. The *value* of the Hamiltonian for a finite region is thus determined by the Hamiltonian boundary term and hence is *quasilocal*: it is associated with the closed spatial 2-surface boundary of the region.

Now we turn to specific details for GR. Because we work with first order Lagrangians, we can easily obtain the Hamiltonian by merely rearranging the Lagrangian into the field theory analogue of $L = p_k \dot{q}^k - H$; essentially from (1) we simply get $\mathcal{L} = P^0_A \partial_t \phi^A - (-P^a_A \partial_a \phi^A + \Lambda)$. We first consider the Hilbert Lagrangian which is given by the spatial integral of the Hilbert Lagrangian density (5). The spatial integrand can be expanded in the form

$$\mathcal{L}_H N^\mu d\Sigma_\mu = \left\{ \dot{\Gamma}^\alpha{}_{\beta\nu} \pi^{\beta\gamma} \delta^{\mu\nu}_{\alpha\gamma} - \mathcal{H}^\mu_H(N) \right\} d\Sigma_\mu, \tag{14}$$

where $d\Sigma_\mu := \frac{1}{3!}\epsilon_{\mu\alpha\beta\gamma} dx^\alpha dx^\beta dx^\gamma$ and our definition of $\dot{\Gamma}^\alpha{}_{\beta\gamma}$, which simply reduces to $N^\mu \partial_\mu \Gamma^\alpha{}_{\beta\gamma} = \partial_t \Gamma^\alpha{}_{\beta\gamma}$ in adapted coordinates, is given in general in appendix A along with

other details regarding our choice of representation. From (5), after a straightforward calculation, without discarding any total derivative term, we obtained the explicit expressions

$$\mathcal{H}^{\mu}{}_{\nu} = -\frac{1}{2} R^{\alpha}{}_{\beta\gamma\lambda} \pi^{\mu\beta} \delta^{\mu\lambda\gamma}_{\alpha\nu\sigma} - \Gamma^{\alpha}{}_{\beta\nu} D_{\gamma} \pi^{\beta\lambda} \delta^{\mu\gamma}_{\alpha\lambda}, \tag{15}$$

$$\mathcal{B}^{\mu\nu}_{H}(N) = N^{\gamma} \pi^{\beta\lambda} \Gamma^{\alpha}{}_{\beta\gamma} \delta^{\mu\nu}_{\alpha\lambda}. \tag{16}$$

Note that, as predicted, the (spatial hyper-) surface density part $\mathcal{H}^{\mu}{}_{\nu}$ vanishes 'on shell', since it is linear in the *non-metricity*, $D_{\alpha} g_{\mu\nu}$, and, with vanishing non-metricity, the curvature contractions reduce to the Einstein tensor, $G_{\mu\nu}$. Hence, as expected, the value of the Hamiltonian comes only from the boundary term $\mathcal{B}$, which gives the quasilocal energy-momentum.

The Hamiltonian symplectic structure can be found, as in the Lagrangian case, by varying the Hamiltonian (regarding it as a function of $\Gamma^{\alpha}{}_{\beta\nu}$ and its conjugate momentum $\pi^{\gamma(\beta} \delta^{\nu)\mu}_{\gamma\alpha} = \delta^{\mu}_{\alpha} \pi^{\beta\nu} - \delta^{(\nu}_{\alpha} \pi^{\beta)\mu}$). This variation fits the general pattern

$$\delta \mathcal{H}^{\mu}(N) = \text{ (field equation terms) } + \partial_{\nu} \mathcal{C}^{\mu\nu}(N). \tag{17}$$

The field equation terms include a set of initial value constraints and dynamical equations [35,1,23] which may be used to calculate the evolution of the gravitational fields. Here our focus, however, is on the variational boundary term $\mathcal{C}^{\mu\nu} = -\mathcal{C}^{\nu\mu}$ which reflects the symplectic structure — and the implicitly built in boundary conditions — of the physical system with respect to the particular Hamiltonian density under consideration. The variation of the spatial hypersurface part, $N^{\mu} \mathcal{H}^{\nu}{}_{\mu}$, in addition to the field equation terms, gives rise to the total divergence

$$\partial_{\tau} [N^{\lambda} (\delta \Gamma^{\alpha}{}_{\beta\mu} \pi^{\beta\sigma} \delta^{\tau\rho\mu}_{\alpha\sigma\lambda} - \Gamma^{\alpha}{}_{\beta\lambda} \delta \pi^{\beta\sigma} \delta^{\tau\rho}_{\alpha\sigma})]. \tag{18}$$

Combining this with the variation of the boundary term (16), we find that, for the present (Hilbert) case the Hamiltonian variational boundary term, $\mathcal{C}$, takes the explicit form

$$\mathcal{C}^{\tau\rho}_{H}(N) = -2\pi^{\beta\nu} \, \delta \Gamma^{\mu}{}_{\alpha\beta} \, \delta^{\alpha[\tau}_{\mu\nu} N^{\rho]}, \tag{19}$$

showing that the 'control variable' is — similar to the Lagrangian case — (certain projected components of) the connection coefficients.

Expanding out the Hamiltonian boundary expression (16), and using the $Dg = 0$ field equation to express the connection coefficients in terms of the metric gives

$$\mathcal{B}^{\mu\nu}_{H}(N) \equiv \kappa^{-1} \sqrt{-g} N^{\gamma} g^{\beta[\nu} \Gamma^{\mu]}{}_{\beta\gamma} \equiv (2\kappa)^{-1} \sqrt{-g} N^{\gamma} g^{\beta\nu} g^{\mu\sigma} (\partial_{\beta} g_{\sigma\gamma} - \partial_{\sigma} g_{\beta\gamma}). \tag{20}$$

This is in fact the *superpotential* which gives rise to the Møller *pseudotensor* [36]. From this calculation we have acquired two insights: first, the Møller pseudotensor is essentially a *quasilocal* object and, second, it really gives the *energy* — the value of the Hamiltonian — for the particular Hamiltonian which generates time displacements in the case that the connection is fixed on the boundary.

However, there are some shortcomings in this Hamiltonian (aside from the obvious fact that the boundary term is not covariant, which we will remedy further below). First, although the boundary condition at infinity is simply $\Gamma = 0$, we must consider the rate of

approach. With the standard fall offs, in particular $\delta\Gamma \to O(1/r^2)$, the boundary term in the variation of the Hilbert Hamiltonian will not automatically vanish asymptotically, indicating that the Hamiltonian is not differentiable on the phase space of all asymptotically flat fields; hence one must actually give the explicit asymptotic functional form for Γ at each instant of time. Second, the actual value of the energy calculated from the boundary term for the Schwartzschild solution *is not* the expected value M but exactly *half* of that amount.

This fact is closely connected with a well-known problematical feature of Komar's covariant expression [33],

$$\mathcal{B}_K^{\tau\rho}(N) := \frac{1}{\kappa}\sqrt{-g}D^{[\tau}N^{\rho]} = \frac{1}{\kappa}\sqrt{-g}(g^{\mu[\tau}\partial_\mu N^{\rho]} + N^\gamma g^{\mu[\tau}\Gamma^{\rho]}{}_{\gamma\mu}), \tag{21}$$

which is equivalent to the Møller superpotential (20) when the components of N^μ are constant. Long ago it was noted that if the Komar expression is normalized to give the correct energy-momentum then it gives a value for the angular momentum which is twice that desired, conversely if it is normalized to give the correct angular momentum it then gives only half of desired amount for the energy. (The proper way to reconcile the Møller-Komar superpotential with the desired energy-momentum and angular momentum results was found some time ago [27,42,16]. In fact the results of those works are forerunners of our preferred expressions discussed below.) Here we found that, from the standard normalization of the Hilbert Lagrangian, the associated boundary term in the Hamiltonian (obtained without discarding or modifying any boundary terms) naturally gives rise to the Møller-Komar superpotential with the latter normalization.

Møller himself later noted that the Hilbert Lagrangian leads to his superpotential [37]. Also, it has long been known that the Hilbert Lagrangian leads via a Noether argument to the Komar superpotential (see, e.g., [4]). (The Komar potential has also been obtained in a Hamiltonian treatment via a Legendre transformation from the Einstein Hamiltonian [16].) Although the factor of 2 problem with Komar's expression has also long been known, yet it seems that only very recent works [24,17] have explicitly noted that the normalization arising directly from the Hilbert Lagrangian gives only half of the expected energy-momentum.

There is a very simple cure to this problem of getting half of the desired value. Exploiting the freedom in the Hamiltonian that we noted above — the freedom to add a divergence free term to the canonical energy-momentum density without changing the fact that it is conserved — we can modify the Hamiltonian boundary term. Adjusting the Hamiltonian boundary term 'by hand' will not change the equations of motion but it will change the boundary conditions and the value of the quasilocal energy-momentum. Note that such an adjustment is essential if we wish to obtain a Hamiltonian which will be differentiable on the phase space of all asymptotically flat fields, as was nicely explained some time ago [40] in connection with the usual ADM formulation. (Indeed the usual approach is simply to discard all boundary terms on the way from the Lagrangian to the Hamiltonian and then to fix up the Hamiltonian boundary term in the end to produce the desired behavior.) Given this freedom, we could simply double the boundary term in the Hilbert Hamiltonian. This would certainly take care of the problem of getting only half the value for the energy, *but* the symplectic structure in the variation of the Hamiltonian would then be modified. The necessary boundary condition would then require the vanishing of

$$\partial_\tau \left\{ N^\lambda \left(\Gamma^\alpha{}_{\beta\lambda}\delta\pi^{\beta\sigma}\delta^{\rho\tau}_{\alpha\sigma} + \delta\Gamma^\alpha{}_{\beta\lambda}\pi^{\beta\sigma}\delta^{\tau\rho}_{\alpha\sigma} + 2\delta\Gamma^\mu{}_{\alpha\beta}\pi^{\beta\nu}\delta^{\alpha[\tau}_{\mu\nu}\delta^{\rho]}_\lambda \right) \right\}, \tag{22}$$

which leads to a rather unattractive, complicated boundary condition requiring the vanishing of a combination of $\delta\pi$ and $\delta\Gamma$ and which, moreover, would not automatically vanish asymptotically — so the Hamiltonian would still not be differentiable on the space of all asymptotically flat fields.

Let us now briefly consider the Einstein Lagrangian density (7). We take the dynamical "coordinate" variable to be $\pi^{\beta\gamma}$. The associated Hamiltonian density can be found from

$$\mathcal{L}_E N^\mu d\Sigma_\mu = \left\{ \dot{\pi}^{\sigma\alpha} \Gamma^\gamma{}_{\sigma\nu} \delta^{\mu\nu}_{\alpha\gamma} - \mathcal{H}^\mu_E(N) \right\} d\Sigma_\mu, \tag{23}$$

(our general definition of $\dot{\pi}$ is given in appendix A; in adapted coordinates it simply reduces to $\partial_t \pi^{\sigma\alpha}$). The Hamiltonian density $\mathcal{H}_E$ still has the same ADM surface part $\mathcal{H}^\mu{}_\nu$ (but when varied it is now to be regarded as a function of $\pi^{\sigma\alpha}$ and its conjugate momentum, $\frac{1}{2}\Gamma^\gamma{}_{\gamma\sigma}\delta^\mu_\alpha + \frac{1}{2}\Gamma^\gamma{}_{\gamma\alpha}\delta^\mu_\sigma - \Gamma^\mu_{\sigma\alpha}$). However the Hamiltonian has now acquired a different boundary term given by

$$\mathcal{B}^{\mu\nu}_E(N) := N^\tau \pi^{\beta\lambda} \Gamma^\alpha{}_{\beta\gamma} \delta^{\mu\nu\gamma}_{\alpha\lambda\tau}. \tag{24}$$

This Hamiltonian boundary term, which arose directly from the Einstein Lagrangian density without discarding or adjusting any exact differentials, is a familiar object. Using the metric compatible field equation to replace the connection by derivatives of the metric leads to a well-known form of the expression,

$$\mathcal{B}^{\mu\nu}_E(N) \equiv (2\kappa) N^\lambda (-g)^{-1/2} g_{\lambda\tau} \partial_\gamma (\pi^{\mu\tau}\pi^{\nu\gamma} - \pi^{\nu\tau}\pi^{\mu\gamma}). \tag{25}$$

This is exactly the Freud superpotential [19] whose divergence gives rise to the Einstein pseudotensor. The spatial integral of the Einstein pseudotensor yields a value which is actually quasilocal, it is given by the integral of the Freud superpotential over the closed 2-boundary of the spatial region. This is identically the same boundary integral and thus the same quasilocal value as is determined by the Einstein Hamiltonian via its boundary term. Extending the region to spatial infinity yields the total energy-momentum, now with the proper normalization.

The boundary term in the variation of the Einstein Hamiltonian takes a form which differs from the Hilbert case:

$$\mathcal{C}^{\tau\rho}_E(N) = 2\Gamma^\mu{}_{\alpha\beta}\, \delta\pi^{\beta\nu} \delta^{\alpha[\tau}_{\mu\nu} N^{\rho]}. \tag{26}$$

From the symplectic structure of this Hamiltonian variation boundary term we learn that this Einstein choice corresponds to holding fixed the contravariant metric density. With the usual asymptotics this term will vanish at spatial (but not at future null) infinity, consequently the Einstein Hamiltonian is automatically differentiable on the phase space of all asymptotically flat fields (spatially, while at future null infinity one must specify the detailed functional asymptotic form of the metric to describe the radiation). The symplectic response, however, reveals a deficiency. Since it is some projected components of the connection, it is not really a covariant object. An improved result could be obtained from the Lagrangian density (9), but we have already seen the important ideas so, instead of elaborating that case, we will just go on to our final forms for the Hamiltonian boundary term in the next section. However before we do that let us make a few observations.

The role of the variational boundary term in the Lagrangian and Hamiltonian formulations are similar, in that in both instances one can adjust the boundary term to change the implicit boundary conditions. However in the Hamiltonian case there is an additional entirely independent and very strong motivation which draws our attention to the boundary term and moreover invites us to modify it. Because the Hamiltonian is conserved, its value has a physical significance not shared by the Lagrangian. This conservation property is preserved under modifications of the boundary term (equivalently, preserved under adding a divergence free term to the Hamiltonian density). In the case of the Hamiltonian for dynamic geometry, the entire value actually comes from the boundary term.

One important way in which the boundary terms in the Lagrangian and Hamiltonian differ is that the former determines a boundary condition on the whole boundary of a space-time region whereas the latter determines only spatial boundary conditions. By adjusting both of them accordingly we can independently choose what is held fixed on the initial time hypersurface and at the spatial boundary (which in fact is convenient for the different types—Cauchy vs. Dirichlet/Neumann—of boundary conditions typically required on these surfaces). This fact is related to another way in which the Hamiltonian boundary term issue differs from that of the Lagrangian. At the Hamiltonian level, we can make boundary terms which depend on the displacement vector field N^μ in various ways. This allows for many more possibilities than those like (7) and (9) that are available at the purely Lagrangian level. Consequently there is a bigger need for a suitably restrictive criterion.

The plain fact is that we can entirely ignore the boundary term which arises from the Lagrangian and simply change the Hamiltonian boundary term to anything we want. However our choice is constrained if we wish to satisfy an important physical desiderata: namely to get the desired energy-momentum values for empty space, weak fields and at spatial and future null infinity. This requirement shows up only at the Hamiltonian level; it is easily dealt with at that level whereas in general it is not so readily satisfied by a judicious adjustment of the boundary term back at the Lagrangian level. In fact this requirement *forces* us to adjust the Hamiltonian boundary term away from that naturally inherited from the Hilbert Lagrangian. Moreover, it actually fixes the form of the Hamiltonian boundary term — but only to linear order. Going beyond the linear order we can use our freedom to build in certain boundary conditions via the Hamiltonian variation symplectic structure.

One consequence of this freedom is that, not only the superpotentials for the Møller and Einstein pseudotensors, but in fact also the superpotentials associated with *all* of the other pseudotensors are likewise acceptable Hamiltonian boundary terms. Here we briefly outline the argument which we have presented in more detail elsewhere [10]. Consider the pseudotensor idea: a suitable *superpotential* $H_\mu{}^{\nu\lambda} \equiv H_\mu{}^{[\nu\lambda]}$ is selected and used to split the Einstein tensor thereby defining the associated gravitational energy-momentum pseudotensor:

$$\kappa\sqrt{-g}N^\mu t_\mu{}^\nu := -N^\mu\sqrt{-g}G_\mu{}^\nu + \frac{1}{2}\partial_\lambda(N^\mu H_\mu{}^{\nu\lambda}), \tag{27}$$

where we have inserted a vector field to make the calculation more nearly covariant. The usual formulation is recovered by taking the components of the vector field to be constant in the present reference frame. Einstein's equation, $G_\mu{}^\nu = \kappa T_\mu{}^\nu$, can now be rearranged into a form where the source is the *total* effective energy-momentum pseudotensor

$$\partial_\lambda H_\mu{}^{\nu\lambda} = 2\kappa\sqrt{-g}\mathcal{T}_\mu{}^\nu := 2\kappa\sqrt{-g}(t_\mu{}^\nu + T_\mu{}^\nu). \tag{28}$$

An immediate consequence of the antisymmetry of the superpotential is that $\mathcal{T}_\mu{}^\nu$ is a conserved current: $\partial_\nu[(-g)^{1/2}\mathcal{T}_\mu{}^\nu] \equiv 0$, which integrates to give a conserved energy-momentum. The energy-momentum within a finite region

$$\begin{aligned} -P(N) &:= -\int_\Sigma N^\mu \mathcal{T}_\mu{}^\nu \sqrt{-g}\, d\Sigma_\nu \\ &\equiv \int_\Sigma [N^\mu\sqrt{-g}(\frac{1}{\kappa}G_\mu{}^\nu - T_\mu{}^\nu) - \frac{1}{2\kappa}\partial_\lambda(N^\mu H_\mu{}^{\nu\lambda})]\, d\Sigma_\nu \\ &\equiv \int_\Sigma N^\mu \mathcal{H}^\nu{}_\mu d\Sigma_\nu + \oint_{S=\partial\Sigma} \mathcal{B}(N) \equiv H(N), \end{aligned} \tag{29}$$

is seen to be just the value of the Hamiltonian. Note that $\mathcal{H}^\nu{}_\mu$ is the covariant form of the ADM Hamiltonian density, which has a vanishing numerical value, so that the value of the Hamiltonian is determined purely by the boundary term $\mathcal{B}(N) = -N^\mu(1/2\kappa)H_\mu{}^{\nu\lambda}(1/2)dS_{\nu\lambda}$. Thus for any pseudotensor the associated *superpotential* is naturally a Hamiltonian boundary term. Moreover the energy-momentum defined by such a pseudotensor does not really depend on the local value of the reference frame, it is actually *quasilocal*—it depends (through the superpotential) on the values of the reference frame (and the fields) only on the boundary of a region.

The Hamiltonian approach endows these quasilocal values with a physical significance. To understand the *physical meaning* of the quasilocalization, calculate the boundary term in the Hamiltonian variation:

$$-\frac{1}{2}\left[\delta\Gamma^\alpha{}_{\beta\lambda}N^\mu\pi^{\beta\sigma}\delta^{\tau\rho\lambda}_{\alpha\sigma\mu} + \frac{1}{2\kappa}\delta(N^\mu H_\mu{}^{\tau\rho})\right] dS_{\tau\rho}. \tag{30}$$

(This result differs slightly from (18) because the ADM form of the Hamiltonian used here does not contain a term proportional to $D\pi$.) For example for the *Einstein* pseudotensor, use the Freud superpotential (20) as the Hamiltonian boundary term in (29). Then the boundary term in the Hamiltonian variation has the integrand $\delta(\pi^{\beta\sigma}N^\mu)\Gamma^\alpha{}_{\beta\lambda}\delta^{\tau\rho\lambda}_{\alpha\sigma\mu}$, which shows not only that $\pi^{\beta\sigma}$ is to be held fixed on the boundary, but also that the appropriate displacement vector field is $N^\mu =$ constant, and the reference configuration here is Minkowski space with a Cartesian reference frame.

A minor variation on the preceding analysis results from choosing a superpotential with a contravariant index: $H^{\mu\nu\lambda} \equiv H^{\mu[\nu\lambda]}$. A further variation: $H^{\mu\nu\alpha} := \partial_\beta H^{\mu\alpha\nu\beta}$, along with the symmetries $H^{\mu\alpha\nu\beta} \equiv H^{\nu\beta\mu\alpha} \equiv H^{[\mu\alpha][\nu\beta]}$ and $H^{\mu[\alpha\nu\beta]} \equiv 0$, leads to a *symmetric* pseudotensor—which then allows for a simple definition of angular momentum, see [35] §20.2. We can cover these options simply by using the displacement vector field to make modifications like $N^\mu H_\mu{}^{\nu\lambda} \longrightarrow N_\mu H^{\mu\nu\lambda}$.

In this way we see that each of the pseudotensors actually gives the value of the quasilocal energy-momentum for an acceptable Hamiltonian. In each case, via the Hamiltonian boundary variation symplectic structure, this quasilocal energy-momentum is associated with some definite physical boundary conditions [10]. Note that this same type of argument extends to superpotentials (i.e., Hamiltonian boundary terms) that are more general than the classic linear-in-displacement form associated with the traditional pseudotensors. In particular one can include first (and even higher) derivatives of the displacement, as occurs in the Komar expression (21).

In summary, similar to the Lagrangian analysis which we have discussed, the boundary term in the Hamiltonian variation, $\mathcal{C}$, generally does not vanish, so the Hamiltonian is not differentiable for general field values. A modification, achieved by adding a total derivative term to the Hamiltonian, adjusts $\mathcal{B}$ without changing the field equations and can compensate, making $\mathcal{C}$ vanish for suitable preselected boundary values. The exact form of such an adjustment still has infinite possibilities, this allows for an infinite number of different gravitational energy definitions [6]. However, each of them has its own unique expression for the Hamiltonian boundary variation $\mathcal{C}$. The symplectic structure of this term reveals the implicit boundary conditions and thereby gives a physical interpretation for each quasilocal energy-momentum expression. Thus, for each well-defined Hamiltonian boundary expression, one can, via the Hamiltonian analysis, find its associated symplectic structure which shows the built in control mode, or equivalently, the implicit boundary conditions.

III. QUASILOCAL ENERGY-MOMENTUM

Here we describe our Hamiltonian boundary term expressions for quasilocal energy-momentum. Our major tool is the symplectic analysis of the Hamiltonian boundary variational principle. We associate each possible Hamiltonian boundary term expression with the boundary conditions identified via the symplectic structure of the boundary term in the variation of the Hamiltonian. There are an infinite number of possible Hamiltonian boundary terms, and correspondingly an infinite number of possible boundary conditions. We greatly reduce this infinity by applying a *covariance* criteria.

In the previous section we saw that the Hilbert Hamiltonian had problems asymptotically while the Einstein Hamiltonian gave good asymptotic values but had a non-covariant response, being linear in the connection. These shortcomings necessitate, as we saw at the Lagrangian level, the introduction of a reference geometry. Hence for regulating the variational boundary term, a background manifold with a suitable geometry, $(\bar{M}, \bar{g}_{\mu\nu}, \bar{\Gamma}^{\alpha}{}_{\mu\nu})$, is introduced as a reference configuration. The gravitational energy-momentum is understood to be measured with respect to this selected background. Any modification of the Lagrangian or Hamiltonian boundary term changes the symplectic structure and the boundary conditions. Here, from the two examples we considered, we obtain modified versions of $\mathcal{B}_H$ and $\mathcal{B}_E$ which have the same the control modes, respectively $\Gamma^{\mu}{}_{\alpha\beta}$ or $\pi^{\mu\nu}$, but their responses are given an improved "covariant" form.

For the metric density (in deference to the traditional choice of variables we refer to it as the "Dirichlet") control mode, the background is just what we need to make the responses become tensorial objects without changing the control variables. Its symplectic structure in the variational boundary term is required to have the form

$$\mathcal{C}^{\tau\rho}_{\pi}(N) = 2\Delta\Gamma^{\mu}{}_{\alpha\beta}\,\delta\pi^{\beta\nu}\delta^{\alpha[\tau}_{\mu\nu}N^{\rho]}, \tag{31}$$

where the Δ means the difference of variables between physical and background configurations (i.e., $\Delta\Gamma^{\mu}{}_{\alpha\beta} := \Gamma^{\mu}{}_{\alpha\beta} - \bar{\Gamma}^{\mu}{}_{\alpha\beta}$ and $\Delta\pi^{\mu\nu} := \pi^{\mu\nu} - \bar{\pi}^{\mu\nu}$). Now the response is a combination of $\Delta\Gamma$ which is a tensor. Moreover the whole Hamiltonian boundary variation term is now the projection along the displacement vector field of a four dimensionally covariant object, a vector density which vanishes asymptotically (spatially) with standard fall offs — showing

that the Hamiltonian is differentiable on the space of all asymptotically flat fields. In order to obtain this desired $\mathcal{C}_\pi$, we must modify $\mathcal{B}_E$. The modified quasilocal energy-momentum boundary term, $\mathcal{B}_\pi$, was found in [12–14] to be

$$\mathcal{B}^{\mu\nu}_\pi(N) = N^\tau \pi^{\beta\lambda} \Delta\Gamma^\alpha{}_{\beta\gamma}\delta^{\mu\nu\gamma}_{\alpha\lambda\tau} + N^\tau \bar{\Gamma}^\alpha{}_{\beta\tau}\Delta\pi^{\beta\lambda}\delta^{\mu\nu}_{\alpha\lambda}. \tag{32}$$

Similarly, for our connection ("Neumann") control mode, the boundary term in the Hamiltonian variation can also be improved by incorporating reference quantities in the form

$$\mathcal{C}^{\tau\rho}_\Gamma = -2\Delta\pi^{\beta\nu}\,\delta\Gamma^\mu{}_{\alpha\beta}\delta^{\alpha[\tau}_{\mu\nu}N^{\rho]}. \tag{33}$$

This symplectic expression is again the projection along the displacement vector field of a four-covariant vector density which automatically vanishes asymptotically (spatially, with standard fall offs) indicating that the Hamiltonian is differentiable on the space of all asymptotically flat fields. This version follows from the adjusted Hamiltonian boundary term

$$\mathcal{B}^{\mu\nu}_\Gamma = N^\tau \bar{\pi}^{\beta\lambda} \Delta\Gamma^\alpha{}_{\beta\gamma}\delta^{\mu\nu\gamma}_{\alpha\lambda\tau} + N^\gamma \Gamma^\alpha{}_{\beta\gamma}\Delta\pi^{\beta\lambda}\delta^{\mu\nu}_{\alpha\lambda}. \tag{34}$$

Note that the two modes are complimentary: the Hamiltonian boundary variation symplectic relation for one can be obtained from the other just by interchanging the control-response roles.

From the variables at hand there are two other Hamiltonian boundary term expressions which can be constructed:

$$\mathcal{B}^{\mu\nu}_0 = N^\tau \bar{\pi}^{\beta\lambda} \Delta\Gamma^\alpha{}_{\beta\gamma}\delta^{\mu\nu\gamma}_{\alpha\lambda\tau} + N^\gamma \bar{\Gamma}^\alpha{}_{\beta\gamma}\Delta\pi^{\beta\lambda}\delta^{\mu\nu}_{\alpha\lambda}, \tag{35}$$

$$\mathcal{B}^{\mu\nu}_1 = N^\tau \pi^{\beta\lambda} \Delta\Gamma^\alpha{}_{\beta\gamma}\delta^{\mu\nu\gamma}_{\alpha\lambda\tau} + N^\gamma \Gamma^\alpha{}_{\beta\gamma}\Delta\pi^{\beta\lambda}\delta^{\mu\nu}_{\alpha\lambda}. \tag{36}$$

$\mathcal{B}_0$ has the interesting property of being linear in the dynamic variables π, Γ while $\mathcal{B}_1$ is linear in $\bar{\pi}$, $\bar{\Gamma}$. The two associated Hamiltonian variation boundary terms (both of which automatically vanish asymptotically with standard spatial fall offs) have a remarkable $\Delta \leftrightarrow \delta$ symmetry:

$$\mathcal{C}^{\tau\rho}_0 = -\delta\Gamma^\alpha{}_{\beta\gamma}\Delta\pi^{\beta\sigma}N^\mu\delta^{\tau\rho\gamma}_{\alpha\sigma\mu} - \Delta\Gamma^\alpha{}_{\beta\gamma}N^\gamma\delta\pi^{\beta\sigma}\delta^{\tau\rho}_{\alpha\sigma}, \tag{37}$$

$$\mathcal{C}^{\tau\rho}_1 = \quad \Delta\Gamma^\alpha{}_{\beta\gamma}\delta\pi^{\beta\sigma}N^\mu\delta^{\tau\rho\gamma}_{\alpha\sigma\mu} + \delta\Gamma^\alpha{}_{\beta\gamma}N^\gamma\Delta\pi^{\beta\sigma}\delta^{\tau\rho}_{\alpha\sigma}. \tag{38}$$

However, neither has the $J^{[\tau}N^{\rho]}$ form of a projection along N^μ of a 4-dimensionally covariant vector density, only our two expressions (32,34) (or constant linear combinations thereof) leading to (31,33) have this desirable 'covariant' property.

Returning to our two 'covariant' expressions, there is a technical hitch here that needs discussion. Although the Hamiltonian variation control-response symplectic structure has a nice covariant form, the Hamiltonian boundary terms themselves (32,34) are not fully covariant. This is an inevitable consequence of our particular style of first order 'independent metric, frame and connection' formulation, as we briefly explain here (the main technical point is that we actually treat the connection as a one form; for further remarks see appendix A). The connection is not a covariant object. The Hamiltonian must generate the evolution of the connection coefficients including the reference frame gauge dependent part (which

depends on the displacement vector field differentially). This latter task is the duty of the $N\Gamma D\pi$ term in the Hamiltonian. The Hamiltonian boundary term then includes an associated piece with the form $N\Gamma\Delta\pi$. The contribution of this piece to the value of the energy-momentum is a mixture of a covariant physical contribution along with an energy-momentum associated with the particular reference frame. Fortunately these contributions can easily be separated by using the identity

$$N^\mu \Gamma^\alpha{}_{\beta\mu} \equiv D_\beta N^\alpha - \partial_\beta N^\alpha, \tag{39}$$

to replace the $N\Gamma$ terms. The ∂N terms produce a noncovariant (reference frame dependent) unphysical contribution (which can usually be made to vanish in a specially selected frame) and should be dropped (for the purposes of calculating physical energy-momentum but not for calculating the evolution equations). This leads to the final *fully covariant* form of our Hamiltonian boundary quasilocal energy-momentum expressions:

$$\mathcal{B}_\pi^{\mu\nu}(N) = N^\tau \pi^{\beta\lambda} \Delta\Gamma^\alpha{}_{\beta\gamma} \delta^{\mu\nu\gamma}_{\alpha\lambda\tau} + \bar{D}_\beta N^\alpha \Delta\pi^{\beta\lambda} \delta^{\mu\nu}_{\alpha\lambda}, \tag{40}$$
$$\mathcal{B}_\Gamma^{\mu\nu}(N) = N^\tau \bar{\pi}^{\beta\lambda} \Delta\Gamma^\alpha{}_{\beta\gamma} \delta^{\mu\nu\gamma}_{\alpha\lambda\tau} + D_\beta N^\alpha \Delta\pi^{\beta\lambda} \delta^{\mu\nu}_{\alpha\lambda}. \tag{41}$$

We wish to emphasize that an alternate, *fully covariant, direct* derivation of these expressions can be obtained from a different representation as indicated in Appendix A.

After a bi-metric manipulation (see Appendix B), the above expressions can be rewritten in the following compact and remarkably similar forms:

$$\mathcal{B}_\pi^{\mu\nu} = 2\Delta(\pi^{\lambda[\nu} D_\lambda N^{\mu]}) + N^\nu k^\mu(\pi) - N^\mu k^\nu(\pi), \tag{42}$$
$$\mathcal{B}_\Gamma^{\mu\nu} = 2\Delta(\pi^{\lambda[\nu} D_\lambda N^{\mu]}) + N^\nu k^\mu(\bar{\pi}) - N^\mu k^\nu(\bar{\pi}), \tag{43}$$

where

$$k^\mu(\pi) := \pi^{\mu\nu}\Delta\Gamma^\lambda{}_{\nu\lambda} - \pi^{\alpha\beta}\Delta\Gamma^\mu{}_{\alpha\beta}, \tag{44}$$

and $k^\mu(\bar{\pi})$ has the same form with $\bar{\pi}$ replacing π.

At first we were surprised to learn that our Dirichlet expression (40) is *exactly* identical with an expression obtained by Katz *et al.* [34,28,29] which was derived in a completely different way, namely by applying the Noether conservation theorem to the Lagrangian density (compare with (9)):

$$\begin{aligned}\mathcal{L}_\pi &= \pi^{\mu\nu} R_{\mu\nu} + \partial_\mu k^\mu(\pi) - \bar{\pi}^{\mu\nu}\bar{R}_{\mu\nu} \\ &= -\pi^{\mu\nu}(\Delta\Gamma^\lambda{}_{\rho\lambda}\Delta\Gamma^\rho{}_{\mu\nu} - \Delta\Gamma^\lambda{}_{\mu\rho}\Delta\Gamma^\rho{}_{\nu\lambda}) + \Delta\pi^{\mu\nu}\bar{R}_{\mu\nu},\end{aligned} \tag{45}$$

which includes background terms in addition to terms quadratic in the first derivatives of $g_{\mu\nu}$. In retrospect we realize that our having found an identical energy-momentum expression is not so surprising after all. The Hamiltonian approach, as we discussed, is closely connected with the Noether approach. Moreover our covariance requirement leaves little room in the Hamiltonian boundary term for anything else except expressions that can be inherited from a suitable four dimensionally covariant Lagrangian.

Comparing the remarkable similarity in the form of our alternate Neumann expression (41) with that of (40), invites us to consider also obtaining it from a Lagrangian density.

The desired result is obtained simply by interchanging the roles of g and $\bar{g}$ (consequently $\Delta \to -\Delta$) followed by an overall sign change. Thus, we found that (43) can be derived by the same Noether argument used by Katz and coworkers from the following Lagrangian, which is quadratic in the first derivatives of $\bar{g}_{\mu\nu}$:

$$\begin{aligned} \mathcal{L}_\Gamma &= \pi^{\mu\nu} R_{\mu\nu} + \partial_\mu k^\mu(\bar{\pi}) - \bar{\pi}^{\mu\nu} \bar{R}_{\mu\nu} \\ &= -\bar{\pi}^{\mu\nu}(\Delta\Gamma^\lambda{}_{\rho\lambda}\Delta\Gamma^\rho{}_{\mu\nu} - \Delta\Gamma^\lambda{}_{\mu\rho}\Delta\Gamma^\rho{}_{\nu\lambda}) + \Delta\pi^{\mu\nu} R_{\mu\nu}. \end{aligned} \tag{46}$$

Via the Hamiltonian boundary term symplectic structure we identified the two quasilocal energy expressions (40,41) as corresponding to Dirichlet or Neumann type boundary conditions respectively. Comparing their respective Lagrangians we see the relation between these two expressions from a new point of view. There is an amazing symmetry relating the two expressions: one passes into the other simply by interchanging the role of the dynamic physical and background variables. Hence we can regard our "Neumann" expression (41) as giving the energy-momentum of the "reference" geometry measured with respect to the "dynamic" geometry using "Dirichlet" boundary conditions. And, likewise, our "Dirichlet" expression gives the "Neumann" energy-momentum for the "reference" geometry compared to the "dynamic" geometry. This symmetry, however, has an intriguing asymmetry: one might have expected the energy-momentum of the dynamic space referenced to the background to have the same magnitude when the roles are reversed *without* any reversal in the type of boundary condition (on the other hand, one could argue that asymmetries are common when the reference point is changed, e.g., going from 4 to 5 and back to 4 can be described as a 25% increase followed by a 20% decrease). We suspect that there is some, as yet unidentified, underlying principle which could have been used to anticipate this curious symmetry and asymmetry.

Our two boundary expressions are not the only ones for gravitational energy-momentum. They are simply the only ones which satisfy our covariant Hamiltonian symplectic structure criterion. Covariance is a very important property; we believe that a covariant theory should have covariant quasilocal energy-momentum. Nevertheless it is well to keep in mind that some other property may be regarded as even more desirable (also, perhaps our particular implementation of the covariance requirement could be generalized). Then one could exploit the freedom in selecting the Hamiltonian boundary term to achieve a different goal. For example Kijowski and Jezierski [25,31] have used the constraints and the actual boundary conditions required by the field equations to identify and control certain variables representing the true physical degrees of freedom. This necessitates decomposing the dynamic fields into various space, time and boundary components. Consequently their expression for quasilocal energy-momentum is not covariant (although it would be interesting to try to recast it into that form). More recently, following along the lines of Rosenfeld and Belinfante, Petrov and Katz have used what amounts to the same Hamiltonian boundary term freedom that we have exploited to achieve a "symmetric" energy-momentum expression [39]. To any such alternate expressions one can apply, just as we did for the pseudotensors, our Hamiltonian boundary variation symplectic analysis to reveal the implicit spatial boundary conditions.

IV. CONCLUSIONS

In summary, variational principles can yield not only field equations but also boundary conditions; the latter can be modified by adding a total derivative, which is equivalent to a boundary term. The Hamiltonian boundary term for dynamic spacetime governs the value of the Hamiltonian. For each finite region its value yields a quasilocal energy-momentum. The boundary term in the variation of the Hamiltonian has a symplectic structure, which is uniquely determined by the choice of quasilocal expression. Requiring it to vanish associates to each different quasilocal expression distinct boundary conditions. This approach provides a physical interpretation for many of the well-known gravitational energy-momentum expressions including all of the pseudotensors, associating each with unique boundary conditions. Among the infinite possibilities, we found only two Hamiltonian-boundary-term quasilocal expressions which correspond to *covariant* boundary conditions; they are respectively of the Dirichlet or Neumann type. Our Dirichlet expression coincides with the expression recently obtained by Katz and coworkers using Noether arguments and a fixed background. A modification of their argument yields our Neumann expression.

Some key points we have noted in our analysis are:

- The boundary-variation-symplectic-structure principle connects the choice of boundary term with boundary conditions. The Lagrangian boundary term can be adjusted to affect a canonical transformation. It governs the boundary conditions on the 3-dimensional boundary of a spacetime region, including the initial time spacelike hypersurface.

- The Hamiltonian boundary term governs the boundary conditions on the 2-dimensional boundary of the spatial region at each instant of time. The value of the Hamiltonian for dynamic geometry theories including general relativity is determined entirely by the Hamiltonian boundary term. It gives the quasilocal energy-momentum. Our freedom to adjust the Hamiltonian boundary term is justified by the conservation law. The Hamiltonian boundary term depends on the displacement vector, hence it has (in principle) more freedom than is available at the Lagrangian level. However the value of the Hamiltonian, the energy-momentum, also has physical 'correspondence limit' constraints which have no analog for the Lagrangian. The boundary term freedom we exploit here is essentially the same freedom used in constructing 'new improved symmetric energy-momentum tensors'.

- The Einstein and Møller (Komar) pseudotensors arise quite naturally, but the latter has several more shortcomings. All of the pseudotensor superpotentials are possible Hamiltonian boundary terms. Consequently all pseudotensors have quasilocal energy-momentum which is identical to the value of the Hamiltonian for an acceptable choice of boundary term, which, in turn, corresponds to some definite boundary conditions.

- Our 'covariance' criterion removes most of the freedom (leaving only two choices). In hindsight we see that it essentially restricts us to expressions which could be obtained (without any adjustments by hand) by projecting a judicious choice of Lagrangian boundary term. Our Dirichlet mode unexpectedly coincides with that of Katz et al.

In retrospect that is no surprise. Our Neumann mode can be interpreted as the Katz et al. energy-momentum of the reference geometry referred to the dynamic geometry.

We note some features of our quasilocal energy-momentum expressions:

- We found only two "covariant" Hamiltonian boundary expressions. They each give rise to a boundary term in the variation of the Hamiltonian which has the form of a projection of a covariant vector density along the displacement vector field. The form of this variation boundary term shows that the respective Hamiltonians evolve field values with Dirichlet or Neumann type boundary conditions. With standard fall offs, the two Hamiltonians have well defined variational derivatives on the space of asymptotically flat fields at spatial infinity.

- Our expressions depend on a reference configuration, which is required only on the boundary. The reference configuration determines the zero point for all of the quasilocal quantities. The obvious choice is Minkowski space; alternatives which may be more appropriate for certain applications include (anti-)de Sitter space, a Friedmann-Robertson-Walker cosmology, and Schwartzschild geometry. Some options for attaching an appropriate reference configuration to a dynamic boundary were discussed in [14].

- Our expressions also depend on a displacement vector field which selects the associated component of the quasilocal energy-momentum. How to choose the exact form of this vector field was discussed in [14]; the recommended choice is a Killing vector of the reference geometry. In addition to energy-momentum (obtained from a spacetime translation), for a suitable choice of rotational displacement, the expressions also give angular momentum.

- Our expressions reduce to expressions proposed by others in the appropriate limits, in particular to the well known quasilocal expressions of Brown & York [8] and asymptotically to that of Beig & Ó Murchadha [2]. Asymptotically they are equivalent to an expression which gives the expected values at spatial infinity (for asymptotically flat *and* anti-de Sitter solutions) [21]. Moreover, asymptotically, at future null infinity, our Dirichlet expression yields the expected Bondi values [22]. Quasilocally, we have evaluated them for spherically symmetric spacetimes [13,14].

- Katz and coworkers have applied their expression (equivalent to our Dirichlet expression) at future null infinity [29] to cosmology [28] and Mach's principle [34]. We have applied our formulation to black hole thermodynamics [13,14] to obtain the first law and an expression for the entropy.

More generally our work reveals some of the merits of the symplectic Hamiltonian boundary variational principle. In particular it allows us to supplement the usual (correspondence limit to weak field and asymptotic forms) constraints on quasilocal energy-momentum expressions with a principle which connects each quasilocal expression with a distinct boundary condition. Coupled with the covariance criteria the form of the quasilocal energy-momentum expression is then strongly restricted.

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APPENDIX A: DYNAMICAL DETAILS

Our Hamiltonian formalism is adapted to evolving the components of objects including the connection coefficients, hence it includes some (unphysical) dynamic reference frame gauge generation features. There are alternate representations (e.g., [31]) in which these terms do not show up.

Our general approach is to work with a dynamical Lagrangian and Hamiltonian formulation which gives independent equations for evolving the frame, metric and connection, and which handles a wide range of theories, including geometric gravity theories and gauge theories in a uniform way [14]. Note that in such a formalism the Hamiltonian must include the ability to generate a general time dependent (purely gauge) evolution of the frame and the associated induced effects on the components of geometric objects. In general we found that there are certain technical advantages in using a differential form representation. However in the present work we wanted to make our results for the specific case of Einstein's GR more accessible to others, so we transcribed it into the ordinary "holonomic frame" representation. To achieve this some choices must be made: in particular, how to deal with the metric, frame and connection variables and how to impose the vanishing torsion and metric compatible constraints. We want to keep our first order form, so we certainly need the connection and metric to be independent at least to some extent. We elected to impose vanishing torsion 'a priori' and thus to use a symmetric connection and a variational principle which would give the metric compatible condition as a (vacuum) field equation. Because we are using a holonomic frame the evolution of the frame is rather trivial, so we dropped it and its conjugate momentum from our list of dynamic variables. Nevertheless we did not want to depart far from the form of our earlier more general work. Thus the expressions given here are actually obtained by specializing our earlier work. In particular our time derivative is specified by projecting the Lie derivative $\pounds_N := di_N + i_N d$ of *components* of the connection one-form and its conjugate momentum 2-form:

$$\dot{\Gamma}^\alpha{}_{\beta\gamma} dx^\gamma := \pounds_N(\Gamma^\alpha{}_{\beta\lambda} dx^\lambda) = (N^\mu \partial_\mu \Gamma^\alpha{}_{\beta\gamma} + \Gamma^\alpha{}_{\beta\mu} \partial_\gamma N^\mu) dx^\gamma, \tag{A1}$$

$$\dot{\pi}^{\beta\sigma} \epsilon_{\alpha\sigma} := \pounds_N(\pi^{\beta\sigma} \epsilon_{\alpha\sigma}) = (N^\lambda \partial_\lambda \pi^{\beta\sigma} \delta^\rho_\alpha + \frac{1}{2} \pi^{\beta\nu} \partial_\mu N^\lambda \delta^{\mu\rho\sigma}_{\alpha\nu\lambda}) \epsilon_{\rho\sigma}, \tag{A2}$$

where $\epsilon_{\mu\nu} := (1/2)\epsilon_{\mu\nu\alpha\beta} dx^\alpha \wedge dx^\beta$. These "definitions" differ from the usual holonomic expression of the components of the Lie derivative for the contravariant metric density:

$$\pounds_N \pi^{\mu\nu} := \partial_\lambda (N^\lambda \pi^{\mu\nu}) - \pi^{\alpha\nu} \partial_\alpha N^\mu - \pi^{\mu\alpha} \partial_\alpha N^\nu \tag{A3}$$

and the connection coefficients:

$$\pounds_N \Gamma^\alpha{}_{\beta\gamma} := -R^\alpha{}_{\beta\gamma\mu} N^\mu + D_\gamma D_\beta N^\alpha. \tag{A4}$$

The difference, however, shows up only in terms proportional to the derivative of N. (All such terms vanish if the coordinates are adapted so that $\pounds_N$ reduces to ∂_t). A feature of our approach is a frame-gauge generating term in the Hamiltonian density and an associated term in the Hamiltonian boundary quasilocal expression. We like this set up for it is just the way things come out for the vector potential (i.e. connection one-form) in gauge theories like electromagnetism and Yang-Mills.

It is certainly possible to use the usual Lie derivative in a Hamiltonian formulation (see, e.g., [16,32,31]). We choose to avoid it also because it includes an inconvenient for us second derivative of N (which necessitates adjustments in our argument regarding the form of and the vanishing of the Hamiltonian density). A price we pay is then an awkward term like $N^\tau \bar{\Gamma}^\alpha{}_{\beta\tau} \Delta\pi^{\beta\lambda} \delta^{\mu\nu}_{\alpha\lambda}$ in each of our quasilocal energy-momentum Hamiltonian boundary expressions. We have argued that such terms are necessary to give us the Hamiltonian evolution and boundary variation symplectic structure in our representation. Unfortunately the quasilocal energy-momentum, defined as the value of our Hamiltonian, consequently includes both a physical and an unphysical, reference frame gauge dependent, contribution. To separate these effects we rearrange the symmetric connection identity $(\pounds_N e_\beta)^\alpha \equiv [N, e_\beta]^\alpha \equiv (\nabla_N e_\beta - \nabla_\beta N)^\alpha$ to give

$$N^\mu \Gamma^\alpha{}_{\beta\mu} \equiv D_\beta N^\alpha + (\pounds_N e_\beta)^\alpha, \tag{A5}$$

which can be used to replace the $N\Gamma$ factors. The $\pounds_N e_\beta$ term is a non-covariant, dynamic reference frame piece. Its contribution to the energy-momentum can be thought of as an energy associated with the observer. In fact, for any given displacement vector field N, we can choose the reference frame e_β so that it vanishes.

Having introduced this identity, an alternative approach is available. We could treat this term in the same way as its analogue is dealt with in other representations, in particular Kijowski's [31]. Note that, since it includes a time derivative, it really has no place in a Hamiltonian. Rather it should be treated as term belonging to the $p_k \dot{q}^k$ part of the action, a term that shows up in a 2-dimensional integral over the boundary of the spacelike hypersurface rather than in the 3-dimensional hypersurface integral. An easy way to establish the association between this part of our representation and Kijowski's is to consider the frame to be orthonormal. Then its time evolution is just an instantaneous Lorentz boost (in the spacetime 2-plane orthogonal to the spatial boundary) by a hyperbolic angle $\dot{\alpha}\delta t$. The associated 'conjugate momentum' is the area of the 2-surface. Hayward [20] gives another route to time derivative terms on the spatial 2-boundary. He uses the fact that the total boundary term in the Einstein action (7) can be expressed as the extrinsic curvature of the boundary. The standard definition of the extrinsic curvature involves the normal to the boundary surface. But converting the total derivative form to a surface integral is then a delicate task, as the normal is discontinuous on the corners of the usual 3-boundary, which consists of an initial and final constant time spacelike hypersurface connected by a topologically $S^2 \times [t_i, t_f]$ type 3-manifold. This leads to contributions in the action given by the difference between an integral over the final and initial 2-boundary. Contributions which can, in turn, be written as the integral over time of a total time derivative of a 2-boundary term.

Actually it is not difficult to obtain a fully covariant Hamiltonian density with our fully covariant quasilocal boundary terms. Beginning from the Hilbert Lagrangian (5), the

Hamiltonian can be derived by using the usual Lie derivative of a connection (A4)

$$\mathcal{H}^\mu(N) := \pounds_N \Gamma^\alpha{}_{\beta\nu}\pi^{\beta\sigma}\delta^{\mu\nu}_{\alpha\sigma} - N^\mu \mathcal{L}_H \tag{A6}$$

$$\equiv -\frac{1}{2}N^\nu R^\alpha{}_{\beta\gamma\lambda}\pi^{\mu\beta}\delta^{\mu\lambda\gamma}_{\alpha\nu\sigma} - D_\beta N^\alpha \delta^{\mu\nu}_{\alpha\sigma} D_\nu \pi^{\beta\sigma} + \partial_\nu\left(D_\beta N^\alpha \pi^{\beta\sigma}\delta^{\mu\nu}_{\alpha\sigma}\right). \tag{A7}$$

The boundary term here is just the Komar superpotential (with the normalization that gives half of the desired energy-momentum). The boundary term in the variation of this Hamiltonian will not automatically vanish asymptotically; hence this Hamiltonian requires the explicit functional form of the connection to be fixed on the boundary even asymptotically. Consequently this Hamiltonian should be adjusted. Replacing the boundary term by one of the improved boundary terms (40) or (41) gives a fully 4-covariant Hamiltonian for general relativity. Explicitly calculating the resultant boundary term in the variation of the Hamiltonian then leads to the desirable asymptotically well behaved covariant symplectic structures (31), (33). For constant components N^μ these fully covariant Hamiltonian density plus boundary term expressions reduce to (15,16,32,34).

APPENDIX B: GEOMETRY OF BI-METRIC SPACETIME

A background is needed to determine well-defined conserved quantities in GR. For the special choice of mapping and coordinates such that a point P of the physical configuration is mapped into a point $\bar{P}$ of the background and both are given the same coordinates x^μ, the whole system can be looked at as a spacetime M possessing two metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. Geometric quantities with respect to each metric can then be reformulated in terms of the difference between them. In particular each metric determines its own associated Levi-Civita connection and Riemannian geometry.

The simplest case for the connection, from which all others can be derived is

$$\left(D_\mu - \bar{D}_\mu\right) N^\alpha = \Delta\Gamma^\alpha{}_{\mu\nu} N^\nu, \tag{B1}$$

where the variables and operators are denoted with or without a bar consistently with the notation for the metric, and the symbol Δ means the difference of operands between two metrics such as $\Delta\Gamma = \Gamma - \bar{\Gamma}$. This identity shows that $\Delta\Gamma$, being the difference between two connections is a covariant tensorial object.

The Ricci tensor $R_{\mu\nu}$ ($\bar{R}_{\mu\nu}$) with respect to $\Gamma^\alpha_{\beta\mu}$ ($\bar{\Gamma}^\alpha_{\beta\mu}$) can be rewritten, respectively as

$$R_{\mu\nu} = \bar{D}_\lambda \Delta\Gamma^\lambda{}_{\mu\nu} - \bar{D}_\mu \Delta\Gamma^\lambda{}_{\nu\lambda} + \Delta\Gamma^\rho{}_{\mu\nu}\Delta\Gamma^\lambda{}_{\rho\lambda} - \Delta\Gamma^\rho{}_{\mu\lambda}\Delta\Gamma^\lambda{}_{\nu\rho} + \bar{R}_{\mu\nu}, \tag{B2}$$

$$\bar{R}_{\mu\nu} = -D_\lambda \Delta\Gamma^\lambda{}_{\mu\nu} + D_\mu \Delta\Gamma^\lambda{}_{\nu\lambda} + \Delta\Gamma^\rho{}_{\mu\nu}\Delta\Gamma^\lambda{}_{\rho\lambda} - \Delta\Gamma^\rho{}_{\mu\lambda}\Delta\Gamma^\lambda{}_{\nu\rho} + R_{\mu\nu}. \tag{B3}$$

Two other useful identities concern the total derivative terms, which are added to the Hilbert Lagrangian density in order to make the Lagrangian density quadratic in the first order derivatives of the metric:

$$\partial_\mu k^\mu(\pi) = -\pi^{\mu\nu}\Big\{(\bar{D}_\lambda \Delta\Gamma^\lambda{}_{\mu\nu} - \bar{D}_\mu \Delta\Gamma^\lambda{}_{\nu\lambda}) + 2(\Delta\Gamma^\rho{}_{\mu\nu}\Delta\Gamma^\lambda{}_{\rho\lambda} - \Delta\Gamma^\rho{}_{\mu\lambda}\Delta\Gamma^\lambda{}_{\nu\rho})\Big\}, \tag{B4}$$

$$\partial_\mu k^\mu(\bar{\pi}) = -\bar{\pi}^{\mu\nu}(\bar{D}_\lambda \Delta\Gamma^\lambda{}_{\mu\nu} - \bar{D}_\mu \Delta\Gamma^\lambda{}_{\nu\lambda}), \tag{B5}$$

where the $k^\mu(\pi)$ and $k^\mu(\bar{\pi})$ were defined in connection with (44).

REFERENCES

[1] R. Arnowitt, S. Deser and C. W. Misner, "The Dynamics of General Relativity", in *Gravitation: An Introduction to Current Research*, ed. L. Witten (Wiley, New York, 1962) 227–65.

[2] R. Beig and N. ó Murchadha, "The Poincaré Group is the Symmetry Group of Canonical General Relativity", *Ann. Phys.* **174** 463–98 (1987).

[3] F. Belinfante, "On the Current and the Density of the Electric Charge, the Energy, the Linear Momentum, and the Angular Momentum of Arbitrary Fields", *Physica* **7** 449–74 (1940).

[4] P. G. Bergman, "The General Theory of Relativity", in: *Encyclopedia of Physics Vol. IV: Principles of Electrodynamics and Relativity*, ed. S. Flüge, 1962, pp 203–272.

[5] G. Bergqvist, "Quasilocal Mass for Event Horizons", *Class. Quantum Grav.* **9** 1753–68 (1992).

[6] G. Bergqvist, "Positivity and Definitions of Mass", *Class. Quantum Grav.* **9** 1917–22 (1992).

[7] A. Borowiec, M. Ferraris, M. Francaviglia and I. Volovich, "Energy-Momentum Complex for Nonlinear Gravitational Lagrangians in the First-Order Formalism", *Gen. Rel. Grav.* **26** 637–45 (1994).

[8] J. D. Brown and J. W. York, Jr., "Quasilocal Energy and Conserved Charges Derived from the Gravitational Action", *Phys. Rev.* **D47** 1407–19 (1993); `gr-qc/9209012`.

[9] C.-C. Chang, "The Localization of Gravitational Energy: Pseudotensors and Quasilocal Expressions", *MSc. Thesis* (National Central University, Chungli, 1998) unpublished.

[10] C.-C. Chang, J. M. Nester and C.-M. Chen, "Pseudotensors and Quasilocal Energy-Momentum", *Phys. Rev. Lett.* **83** 1897–901 (1999); `gr-qc/9809040`.

[11] C.-C. Chang, J. M. Nester and C.-M. Chen, "Energy-Momentum (Quasi-)Localization for Gravitating Systems", to appear in *Proceedings of the 4-th international workshop on gravitation and astrophysics*, Beijing, October 1999; `gr-qc/9912058`.

[12] C.-M. Chen, "Quasilocal Quantities for Gravity Theories", *MSc. Thesis* (National Central University, Chungli, 1994) unpublished.

[13] C.-M. Chen, J. M. Nester and R.-S. Tung, "Quasilocal Energy-Momentum for Geometric Gravity Theories", Phys. Lett. **203A** 5–11 (1995); `gr-qc/9411048`.

[14] C.-M. Chen and J. M. Nester, "Quasilocal Quantities for General Relativity and other Gravity Theories", *Class. Quantum Grav.* **16** 1279–304 (1999); `gr-qc/9809020`.

[15] D. Christodoulou and S.-T. Yau, "Some Remarks on the Quasi-local Mass", in *Mathematics and General Relativity*, ed J. A. Isenberg (American Mathematical Society, Providence, 1988) 9–16.

[16] P. T. Chruściel, "On the Relation between the Einstein and the Komar Expressions for the Energy of the Gravitational Field", *Ann. Inst. H. Poincaré* **42** 267–82 (1985).

[17] P. T. Chruściel, J. Jezierski and M. A. H. MacCallum, "Uniqueness of the Trautman-Bondi Mass", *Phys. Rev.* **D 58** 084001 (1998).

[18] M. Ferraris and M. Francaviglia, "Covariant First-Order Lagrangians, Energy-Density and Superpotentials in General Relativity", *Gen. Rel. Grav.* **22** 965–85 (1990).

[19] P. Freud, "Über die Ausdrücke der Gesamtenergie und des Gesamt-Impulses eines Materiellen Systems in der Allgemeinen Relativitätstheorie", *Ann. Math.* **40** 417–9 (1939).

[20] G. Hayward, "Gravitational Action for Spacetimes with Nonsmooth Boundaries", *Phys. Rev.* **D 47** 3275–80 (1993).

[21] R. D. Hecht and J. M. Nester, "A New Evaluation of PGT Mass and Spin", *Phys. Lett.* **A 180** 324–31 (1993).

[22] R. D. Hecht and J. M. Nester, "An Evaluation of Mass and Spin at Null Infinity for the PGT and GR Gravity Theories", *Phys. Lett.* **A 217** 81–9 (1996).

[23] J. Isenberg and J. M. Nester, "Canonical Gravity", in *General Relativity and Gravitation: One Hundred Years after the Birth of Albert Einstein, Vol 1.* ed A. Held (Plenum, New York, 1980) pp 23–77.

[24] V. Iyer and R. M. Wald, "Some Properties of the Noether Charge and a Proposal for Dynamical Black Hole Entropy", *Phys. Rev.* **D 50** 846–64 (1994).

[25] J. Jezierski and J. Kijowski, "The Localization of Energy in Gauge Field Theories and in Linear Gravitation", *Gen. Rel. Grav.* **22** 1283–307 (1990).

[26] B. Julia and S. Silva, "Currents and Superpotentials in Classical Gauge-Invariant Theories: I. Local Results with Applications to Perfect Fluids and General Relativity", *Class. Quantum Grav.* **15** 2173–215 (1998).

[27] J. Katz, "A Note on Komar's Anomalous Factor", *Class. Quantum Grav.* **2** 423–5 (1985); J. Katz and A. Ori, "Localization of Field Energy", *Class. Quantum Grav.* **7** 787–802 (1990).

[28] J. Katz, J. Bičák and D. Lynden-Bell, "Relativistic Conservation Laws and Integral Constraints for Large Cosmological Perturbations", *Phys. Rev.* **D 55** 5957–69 (1997).

[29] J. Katz and D. Lerer, "On Global Conservation Laws at Null Infinity", Class. Quantum Grav. **14** 2249–66 (1997); `gr-qc/9612025`.

[30] J. Kijowski, "Asymptotic Degrees of Freedom and Gravitational Energy", in *Proc. Journées Relativistes (Torino, 1983)*, S. Benenti and M. Francaviglia, eds. (Pitagora Edrice, Bologna, 1985) pp 205–219.

[31] J. Kijowski, "A Simple Derivation of Canonical Structure and Quasi-local Hamiltonians in General Relativity", *Gen. Rel. Grav.* **29** 307–43 (1997).

[32] J. Kijowski and W. M. Tulczyjew, *A Symplectic Framework for Field Theories*, (Lecture Notes in Physics, Vol.107, Springer-Verlag, Berlin, 1979).

[33] A. Komar, "Covariant Conservation Laws in General Relativity", *Phys. Rev.* **113** 934–6 (1959).

[34] D. Lynden-Bell, J. Katz and J. Bičák, "Mach's Principle from the Relativistic Constraint Equations", *Mon. Not. R. Astron. Soc.* **272** 150–60 (1995), Erratum: *ibid* **277** (1995) 1600.

[35] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (Freeman, San Francisco, 1973).

[36] C. Møller, "On the Localization of the Energy of a Physical System in the General Theory of Relativity", *Ann. Phys.* **4** 347–71 (1958).

[37] C. Møller, "Further Remarks on the Localization of the Energy in the General Theory of Relativity", *Ann. Phys.* **12** 118–33 (1961).

[38] R. Penrose, "Quasi-local Mass and Angular Momentum in General Relativity", *Proc. R. Soc.* **A381** 53–63 (1982).

[39] A. N. Petrov and J. Katz, "Conservation Laws for Large Perturbations on General Backgrounds", `gr-qc/9905088`; "Relativistic Conservation Laws on Curved Backgrounds

and the Theory of Cosmological Perturbations", `gr-qc/9911025`.

[40] T. Regge and C. Teitelboim, "Role of Surface Integrals in the Hamiltonian Formulation of General Relativity", *Ann. Phys.* **88** 286–319 (1974).

[41] L. Rosenfeld, "Sur le tenseur d'impulsion-énergy", *Acad. R. Belg. Mém. Class. Sci.* **18**, No. 6 (1940).

[42] R. D. Sorkin, "Conserved Quantities as Action Variations", in *Mathematics and General Relativity*, ed J. A. Isenberg (American Mathematical Society, Providence, 1988) pp 23–37.

[43] A. Trautman, "Conservation Laws in General Relativity", in *Gravitation, An Introduction to Current Research*, ed. L. Witten (Wiley, New York, 1962), pp 169–198.

[44] R. M. Wald, *General Relativity*, (University of Chicago Press, Chicago, 1984).

[45] R. M. Wald, "Black Hole Entropy is Noether Charge", *Phys. Rev. D* **48** 3427–31 (1993); `gr-qc/9307038`.